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Holomorphic cylinders with Lagrangian boundaries and periodic orbits of Hamiltonian systems

Daniel Gatien [1]

Département de Mathématiques, Université du Québec à Montréal, C.P. 8880, Succursale Centre-Ville, Montréal, Québec, Canada H3C 3P8

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Abstract

This paper outlines how to use holomorphic cylinders with Lagrangian boundaries to prove existence results for periodic orbits of Hamiltonian systems. We describe the case of the cotangent bundle of the Klein bottle, where these results lead to a new obstruction to the existence of Lagrangian embeddings of the Klein bottle into $\mathbb{C}^2$.

Keywords: Hamiltonian system; Holomorphic cylinder; Lagrangian embedding; Klein bottle

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In their 1992 paper [4], Hofer and Viterbo show how to use J-holomorphic curves to obtain existence results for periodic solutions of Hamiltonian systems. Their main theorem may be paraphrased, somewhat imprecisely, as follows: Let (V, ω) be a *rational* compact symplectic manifold, i.e., the homomorphism $\pi_2(V) \to \mathbb{R}$, $\alpha \mapsto \int_\alpha \omega$, has discrete image, and let $\alpha \in [S^2; V]$ be a free homotopy class of least positive symplectic area $\int_\alpha \omega$. Let Σ_0, Σ_∞ be disjoint closed submanifolds of V. For each $J \in \mathcal{J}(V, \omega)$, the set of ω-compatible almost complex structures on V, consider the moduli space of unparametrized J-holomorphic spheres in V that are in the class α and meet both Σ_0 and Σ_∞. It can be shown that, for a dense subset of $\mathcal{J}(V, \omega)$, this moduli space is a C^∞ compact manifold whose cobordism class is independent of J. Assuming this cobordism class does not vanish, Hofer and Viterbo then show that for any Hamiltonian $H : V \to \mathbb{R}$

[1] Current address: Department of Mathematics & Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4K1. E-mail: gatien@math.mcmaster.ca.

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verifying a certain separation property relative to the submanifolds Σ_0 and Σ_∞, the associated Hamiltonian system

$$\dot{x} = X_H(x)$$

has at least one nonconstant periodic solution. (As usual, the symplectic gradient X_H is given by $\mathrm{d}H(x) = \omega(\cdot, X_H(x))$, $x \in V$.)

The J-holomorphic spheres in this theorem are solutions $u : S^2 \to V$ of the Cauchy–Riemann equation

$$\mathrm{d}u + J \circ \mathrm{d}u \circ j = 0,$$

where j is the standard almost complex structure on S^2. Hofer and Viterbo perturb this equation by introducing a new term which depends on the given Hamiltonian H. Their result then follows from a detailed study of the solutions of the perturbed equation that are in the class α. In this announcement, we will outline how Hofer and Viterbo's arguments can be substantially simplified by assuming the submanifolds Σ_0, Σ_∞ are Lagrangian and using holomorphic cylinders with boundaries in $\Sigma_0 \cup \Sigma_\infty$ instead of spheres. (Full details appear in [2].)

Two properties of cylinders largely account for these simplifications. First, because the cylinder $C = [0,1] \times S^1$ is parallelizable, the Cauchy–Riemann equation for maps $u : C \to V$ has a direct interpretation in terms of vector fields on C, and perturbing this equation to introduce the Hamiltonian term is straightforward. (The analogous perturbation for spheres is trickier to set up and later requires more delicate analytical arguments.) Second, because the group of biholomorphisms of C, for any choice of complex structure, is just S^1, the passage from solutions of the Cauchy–Riemann equation to unparametrized J-curves is simpler for cylinders than for spheres, where the corresponding group is $\mathrm{PSL}(2, \mathbb{C})$.

The use of cylinders does, however, add a new wrinkle: whereas the sphere S^2 has a unique complex structure, the space of complex structures on the cylinder is homeomorphic to $\mathbb{R}$. As we will see, an important property of the perturbed equation, largely obscured in the approach based on spheres, is most naturally expressed in terms of complex structures on C. (The new boundary phenomena that also arise present no further difficulties.)

For the purposes of this announcement, we will describe this simplified approach only as it applies to a single example, the cotangent bundle T^*K^2 of the Klein bottle K^2, where the analog for cylinders of the above cobordism condition is met. As is well known, this manifold arises whenever we are given a Lagrangian embedding of the Klein bottle in an arbitrary symplectic manifold: by Weinstein's Lagrangian neighbourhood theorem, a neighbourhood of the embedded K^2 is symplectomorphic to a neighbourhood of the zero section in T^*K^2. Thus our main theorem, stated below, essentially describes a local property of Lagrangian Klein bottles. In case the ambient symplectic manifold is $\mathbb{C}^2$, a global version of the theorem can also be formulated. As we will see in the final section, this leads to a new obstruction to the existence of a Lagrangian embedding of K^2 into $\mathbb{C}^2$.

1. Statement of the main theorem

Identify the Klein bottle K^2 with the quotient of $\mathbb{R}^2$ by the discrete group whose generators are the diffeomorphisms $(q_1, q_2) \mapsto (q_1 + 1, -q_2)$ and $(q_1, q_2) \mapsto (q_1, q_2 + 1)$. Here, q_1, q_2 denote the standard coordinates in $\mathbb{R}^2$; they give rise to local coordinates in K^2 which will also be denoted q_1, q_2. The standard symplectic form ω on T^*K^2 is then given locally by $\omega = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2$, where $p_i(\mathrm{d}q_j) = \delta_{ij}$. Notice that $\mathrm{d}q_1$ is the coordinate expression of a globally defined closed 1-form on K^2. Viewed as a mapping from K^2 to T^*K^2, this 1-form embeds the Klein bottle as a Lagrangian submanifold L_1 of T^*K^2. Similarly, the (image of the) zero section $K^2 \to T^*K^2$ is another embedded Lagrangian Klein bottle L_0 in T^*K^2, and L_0 and L_1 are disjoint. (L_1 is a translate of L_0 in the direction of p_1.) Let γ_0 be the closed curve $\gamma_0(t) = (q_1(t), q_2(t)) = (t, 0)$ in $L_0 = K^2$. Our main result asserts that if $H : T^*K^2 \to \mathbb{R}$ always assumes larger values on L_1 than on L_0, the associated Hamiltonian system has at least one periodic solution. More precisely:

Theorem 1. *Suppose $H : T^*K^2 \to \mathbb{R}$ is a smooth Hamiltonian such that* $\mathrm{d}H$ *is compactly supported and*

$$h = \inf_{L_1} H - \sup_{L_0} H$$

is strictly positive. Then the Hamiltonian system

$$\dot{x} = X_H(x)$$

has a nonconstant T-periodic solution with $T \leqslant 1/h$ and which is freely homotopic to γ_0 in T^*K^2.

2. The perturbed Cauchy–Riemann equation

In analogy with [4], the proof of Theorem 1 considers cylinders in T^*K^2 that are solutions of a perturbed Cauchy–Riemann equation in a given homotopy class. The relevant almost complex structures, variable at the source but fixed in the target, will be described next.

Every Riemann surface homeomorphic to $[0, 1] \times S^1$ is biholomorphic to exactly one compact cylinder $C_\sigma = [0, \sigma] \times S^1 = [0, \sigma] \times \mathbb{R}/\mathbb{Z}$ with $\sigma > 0$, where C_σ inherits its complex structure from $[0, \sigma] \times \mathbb{R} \subset \mathbb{C}$ via the obvious $\mathbb{Z}$-action on the second coordinate. Equivalently, the space of complex structures on the unit cylinder $C = [0, 1] \times \mathbb{R}/\mathbb{Z}$ is parametrized by σ, the associated almost complex structure being

$$\frac{\partial}{\partial s} \mapsto \sigma \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t} \mapsto -\frac{1}{\sigma}\frac{\partial}{\partial s},$$

where $s \in [0, 1]$, $t \in \mathbb{R}/\mathbb{Z}$ are the usual coordinates on C. This almost complex structure, which will also be denoted σ, is simply the pullback via the horizontal rescaling $C \to C_\sigma$, $(s, t) \mapsto (\sigma s, t)$, of the almost complex structure on C_σ. In the sequel, we will usually assume cylinders in T^*K^2 are given as maps on some C_σ, and use (C,σ) only when it is necessary to fix the domain of a family of maps. C_σ will be assumed to carry its standard conformal metric $\mathrm{d}s^2+\mathrm{d}t^2$.

For the target manifold T^*K^2, there is a canonical choice of almost complex structure, for T^*K^2 has a natural Kähler structure. Indeed, by differentiating the above generators of the deck transformation group of $\mathbb{R}^2\to K^2$, we can identify T^*K^2 with the quotient of $\mathbb{C}^2$ by the group

$$G=\bigl\langle (z_1,z_2)\mapsto (z_1+\mathrm{i},-z_2),\ (z_1,z_2)\mapsto (z_1,z_2+\mathrm{i})\bigr\rangle,$$

where $z_k=p_k+\mathrm{i}q_k,\ k=1,2$. Since G is a group of Kähler isometries, T^*K^2 inherits a Kähler structure from $\mathbb{C}^2$ whose associated Kähler form clearly equals the symplectic form ω. We will henceforth let i and $\langle\cdot,\cdot\rangle$ denote the almost complex structure and Riemannian metric on T^*K^2 induced by the standard structures on $\mathbb{C}^2$ via the covering map $\pi:\mathbb{C}^2\to T^*K^2$. Thus $\langle\cdot,\cdot\rangle=\omega(\cdot,\mathrm{i}\cdot)$ and the symplectic gradient X_H of any smooth function $H:T^*K^2\to\mathbb{R}$ is simply i times the $\langle\cdot,\cdot\rangle$ gradient ∇H.

Recalling the definitions of $L_0,L_1\subset T^*K^2$, we now observe that

$$\pi^{-1}(L_0)=\bigl\{(z_1,z_2)\colon\ \operatorname{Re} z_1=\operatorname{Re} z_2=0\bigr\},$$

$$\pi^{-1}(L_1)=\bigl\{(z_1,z_2)\colon\ \operatorname{Re} z_1=1,\ \operatorname{Re} z_2=0\bigr\},$$

so the inclusion $z\mapsto(z,0)$ of $[0,1]\times\mathbb{R}$ into $\mathbb{C}^2$ covers a holomorphic cylinder $u_0:C_1\to T^*K^2$ with boundary in $L=L_0\cup L_1$. In what follows, we will be dealing exclusively with maps $u:(C_\sigma,\partial C_\sigma)\to(T^*K^2,L)$ which, after rescaling horizontally, are homotopic to u_0 in $[C,\partial C;T^*K^2,L]$, the set of homotopy classes of maps $(C,\partial C)\to(T^*K^2,L)$. One reason for this is that we require a uniform bound on the symplectic area: since L is Lagrangian, Stokes's theorem shows that every cylinder in this class has symplectic area equal to $\int u_0^*\omega=1$. Another reason, more closely related to the specific choice of homotopy class, will emerge later.

Relative to the above almost complex structures, the Cauchy–Riemann operator $\bar\partial$ for maps $u:C_\sigma\to T^*K^2$ is given by

$$\bar\partial u=\frac{\partial u}{\partial s}+\mathrm{i}\frac{\partial u}{\partial t},$$

which is a section of the vector bundle $u^*TV\to C_\sigma,\ V=T^*K^2$. Letting $H:T^*K^2\to\mathbb{R}$ be any smooth Hamiltonian satisfying the hypotheses of Theorem 1, we now consider the family of elliptic first order equations

$$\bar\partial u+\lambda\nabla H(u)=0,$$

where $\lambda\in\mathbb{R}$. Our aim is to study C^∞ solutions $u:(C_\sigma,\partial C_\sigma)\to(T^*K^2,L)$ of this equation for which $\lambda\geqslant 0$ and u is homotopic to u_0 in $[C,\partial C;T^*K^2,L]$. Let C denote the set of all such triples (u, λ, σ). Theorem 1 is a straightforward consequence of the following three assertions:

(a) For any solution $(u, \lambda, \sigma) \in \mathcal{C}$, we have

$$\frac{1}{\sigma} \leqslant \int_{C_\sigma} \left|\frac{\partial u}{\partial s}\right|^2 \leqslant 1 - \lambda h,$$

where $h = \inf_{L_1} H - \sup_{L_0} H > 0$. In particular, $\lambda < 1/h$ and $\sigma \geqslant 1$. Moreover, each loop $u(s, \cdot): S^1 \to T^*K^2$ has length at least 1.

(b) Every cylinder in $\mathcal{C}$ is contained in a fixed compact subset of T^*K^2 and has uniformly bounded derivatives of all orders. In other words, there exist constants $c_k > 0$ such that

$$\|u\|_{C^k} \leqslant c_k, \quad k = 0, 1, \ldots,$$

for every $(u, \lambda, \sigma) \in \mathcal{C}$.

(c) There exists a sequence $(u_k, \lambda_k, \sigma_k)$ in $\mathcal{C}$ with $\sigma_k \to \infty$.

To see why Theorem 1 follows, let $(u_k, \lambda_k, \sigma_k)$ be any sequence in $\mathcal{C}$ with $\sigma_k \to \infty$. By passing to a subsequence, we can assume $\lambda_k \to \lambda \in [0, 1/h]$. Then

$$\begin{aligned} 1 &\geqslant \int_{C_{\sigma_k}} \left|\frac{\partial u_k}{\partial s}\right|^2 = \int_0^{\sigma_k}\int_0^1 \left|-\mathrm{i}\frac{\partial u_k}{\partial t} - \lambda_k \nabla H(u_k)\right|^2 \mathrm{d}t\,\mathrm{d}s \\ &= \sigma_k \int_0^1 \left|-\mathrm{i}\frac{\partial u_k}{\partial t}(s_k, t) - \lambda_k \nabla H\big(u_k(s_k, t)\big)\right|^2 \mathrm{d}t, \quad \text{where } 0 \leqslant s_k \leqslant \sigma_k, \\ &= \sigma_k \int_0^1 \left|-\mathrm{i}\dot{x}_k - \lambda_k \nabla H(x_k)\right|^2 \mathrm{d}t, \end{aligned}$$

where $x_k = u_k(s_k, \cdot): S^1 \to T^*K^2$. By assertion (b) and Ascoli's theorem, we can assume $x_k \to x$ in $C^\infty(S^1, T^*K^2)$. The above inequality then shows that $-\mathrm{i}\dot{x} - \lambda\nabla H(x) = 0$, i.e., $\dot{x} = \lambda X_H(x)$. Now, each x_k is freely homotopic to the geodesic γ_0 and has length at least 1, so x also has these properties. In particular, x is nonconstant, and so $\lambda > 0$. The reparametrized loop $t \mapsto x(t/\lambda)$ is then the desired T-periodic solution, where $T \leqslant \lambda \leqslant 1/h$.

3. Why $\mathcal{C}$ contains solutions with σ arbitrarily large

Property (c) asserts that σ, when viewed as a complex structure on the unit cylinder $C = [0, 1] \times \mathbb{R}/\mathbb{Z}$, tends to infinity in the moduli space of all such structures. The proof is based on a cobordism argument whose key steps will now be outlined.

Until further notice, all maps into T^*K^2 will explicitly be defined on the unit cylinder C with its usual flat metric and variable complex structure σ. Translating our previous discussion to this new setting, we find that the set $\mathcal{C}'$ of rescaled solutions consists of all (u,λ,σ) with $u:(C,\partial C)\to(T^*K^2,L)$ homotopic to u_0 and satisfying

$$\overline{\partial}_\sigma u+\lambda\nabla H(u)=0, \tag{1}$$

where

$$\overline{\partial}_\sigma=\frac{1}{\sigma}\frac{\partial}{\partial s}+\mathrm{i}\frac{\partial}{\partial t}$$

is the Cauchy–Riemann operator relative to the structures σ on C and i on T^*K^2. Notice that for $(u,\lambda,\sigma)\in\mathcal{C}'$, we still have $\lambda<1/h$ and $\sigma\geqslant 1$, but, because the metric on C is kept fixed, assertion (b) only ensures the existence of derivative bounds for subsets of $\mathcal{C}'$ with σ bounded. To prove (c), it therefore suffices, by Ascoli's theorem, to show that $\mathcal{C}'$ is noncompact.

This can be achieved by introducing a global setting for the more general inhomogeneous Cauchy–Riemann equation

$$\overline{\partial}_\sigma u+f(z,u)=0, \tag{2}$$

where f is a vector field tangent to $V=T^*K^2$ along the projection $\mathrm{pr}_2:C\times V\to V$ on the second factor, i.e., a section of the bundle $\mathrm{pr}_2^*\,TV\to C\times V$. (Notice that since ∇H can be pulled back to a section of $\mathrm{pr}_2^*\,TV$, (1) is indeed a special case of (2).) This approach essentially dates back to Gromov [3] and was formalized, in the case of disks with Lagrangian boundaries, in [1]. Similar arguments show that, for appropriate choices of a Banach space $\mathcal{F}$ of sections of $\mathrm{pr}_2^*\,TV$ and a manifold $\mathcal{X}$ of (Sobolev or Hölder) maps $u:(C,\partial C)\to(V,L)$ homotopic to u_0, the set

$$\mathcal{M}=\big\{(u,f,\sigma)\in\mathcal{X}\times\mathcal{F}\times(0,\infty):\ \overline{\partial}_\sigma u+f(z,u)=0\big\}$$

is a Banach manifold and the projection $P:\mathcal{M}\to\mathcal{F}$ on the second factor is a Fredholm map. Moreover, it can be shown that 0 is a regular value of P, so $P^{-1}(0)$ is a submanifold of $\mathcal{M}$ of dimension index(P).

The manifold $P^{-1}(0)$ is, in fact, a circle. Indeed, a straightforward argument based on the maximum principle for harmonic functions shows that:

(d) If $u:(C,\partial C)\to(V,L)$ is (σ,i)-holomorphic and homotopic to u_0 in $[C,\partial C;V,L]$, then $\sigma=1$ and $u=u_0$ up to a rotation of C.

We will see in a moment that, by imposing an additional (closed) condition on the elements of $\mathcal{X}$, we can actually assume $P^{-1}(0)=\{(u_0,0,1)\}$. The desired noncompactness of $\mathcal{C}'$ then follows from a simple cobordism argument: Let $\ell\subset\mathcal{F}$ be the closed line segment joining 0 to $(1/h)\nabla H$. Since the preimage $P^{-1}(\ell)$ of ℓ consists of those elements of $\mathcal{C}'$ that verify the new condition, it suffices to show $P^{-1}(\ell)$ is noncompact. Let us assume the opposite and derive a contradiction. Then, because Fredholm maps are locally proper, there exist open neighborhoods $\mathcal{U}$ of $P^{-1}(\ell)$ and $\mathcal{V}$ of ℓ such that $P^{-1}(K)\cap\mathcal{U}$ is compact whenever $K\subset\mathcal{V}$ is compact. Now observe that both endpoints of ℓ are regular values of P, since $P^{-1}((1/h)\nabla H)=\emptyset$. It follows from Smale's generalization of Sard's theorem that ℓ can be perturbed in $\mathcal{V}$, while keeping its endpoints fixed, to be transverse to P. The preimage $P^{-1}(\ell')$ of the resulting transverse curve ℓ' is then a one-dimensional submanifold of $\mathcal{M}$ with boundary $\partial P^{-1}(\ell') = \{(u_0, 0, 1)\}$. The same is of course true of $P^{-1}(\ell') \cap \mathcal{U}$. But since the latter is compact, it defines a cobordism between the singleton $P^{-1}(0)$ and $P^{-1}((1/h)\nabla H) = \emptyset$, which is a contradiction.

It remains to see what condition on the elements of $\mathcal{X}$ will make this a valid argument. In passing from $\mathcal{X}$ to the desired subset $\mathcal{X}'$ (and hence from $\mathcal{M}$ to, say, $\mathcal{M}'$), we require that P restrict to a map whose linearization at $(u_0, 0, 1)$ has no kernel, thus reducing the index from one to zero while maintaining 0 a regular value. In other words, we wish $\mathcal{M}' \subset \mathcal{M}$ to be transverse to the circle $P^{-1}(0)$ at $(u_0, 0, 1)$. Clearly, this can be achieved by choosing in $L_0 = K^2$ a closed curve transverse to γ_0, for example, the curve γ_1 covered by the q_2-axis in $\mathbb{R}^2 = \widetilde{K^2}$, and letting $\mathcal{X}'$ be the set of $u \in \mathcal{X}$ for which $u(0) \in \gamma_1$ (here, "0" denotes the image in C of the origin in $\widetilde{C} = [0, 1] \times \mathbb{R}$).

We conclude this section by explaining why one might expect u_0 to be (essentially) the only holomorphic cylinder in its homotopy class. Identify TK^2 with T^*K^2 in the usual way (so TK^2 is a complex manifold), and introduce a complex structure on the tangent bundle $T\mathbb{R}$ of $\mathbb{R}$ by identifying $s(\mathrm{d}/\mathrm{d}t) \in T_t\mathbb{R}$ with $s + it \in \mathbb{C}$. Then it is easy to check that for any geodesic $\gamma : \mathbb{R} \to K^2$, the differential $\mathrm{d}\gamma : T\mathbb{R} \to TK^2$ is a holomorphic map. In particular, every closed geodesic in K^2 gives rise to a holomorphic (infinite) cylinder in $T^*K^2 = TK^2$. Notice that the cylinder u_0 arises in precisely this way: it is the differential of the closed geodesic $\gamma_0(t) = u_0(0, t)$ in $L_0 = K^2$. Now if two closed geodesics in K^2 are homotopic, so are the resulting holomorphic cylinders. Since γ_0 is in fact unique (among geodesics) in its homotopy class, one might well suspect the same to be true of the cylinder u_0. Assertion (d) confirms this suspicion.

4. Lagrangian embeddings of the Klein bottle

It is still an open question whether or not the Klein bottle admits a Lagrangian embedding into $\mathbb{C}^2$, although a number of obstructions to the existence of such embeddings are known; see [5]. The following global version of Theorem 1 will allow us to add to the list of known obstructions.

Theorem 2. *Let L be a Lagrangian Klein bottle in $\mathbb{C}^2$. Suppose W is a Weinstein neighbourhood of L in $\mathbb{C}^2$ and L' is a translate of L in the direction of the p_1 coordinate in W (just as L_1 was previously obtained from L_0 in T^*K^2). If L' is sufficiently close to L, then for every smooth Hamiltonian $H : \mathbb{C}^2 \to \mathbb{R}$ such that H is linear outside a compact set and*

$$\inf_{L'} H > \sup_L H,$$

the Hamiltonian system $\dot{x} = X_H(x)$ has a (possibly degenerate) periodic solution.

It is likely that a nonconstant periodic solution exists. In either case, however, it follows that L must be *symplectically knotted*, in the sense that L and $\phi(L')$ cannot lie on opposite sides of a hyperplane $P \subset \mathbb{C}^2$ whenever ϕ is a compactly supported symplectomorphism of $\mathbb{C}^2$ with $\operatorname{supp}(\phi) \cap L = \emptyset$. For suppose a separating hyperplane

P exists for some such ϕ. Then $P = f^{-1}(0)$ for some linear function $f:\mathbb{C}^2 \to \mathbb{R}$ with $f|L < 0$ and $f|\phi(L') > 0$. The Hamiltonian system associated to f clearly has no periodic orbits (not even degenerate ones), and hence neither does the system associated to $H = \phi^* f = f \circ \phi$. But since H satisfies the hypotheses of the above theorem, this is a contradiction. The proof of Theorem 2, together with a fuller discussion of symplectic knottedness, appears in [2].

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